The tree of a Non-Archimedean hyperbolic plane

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Start with ordered field $\mathbb{R}$

hyperbolic plane

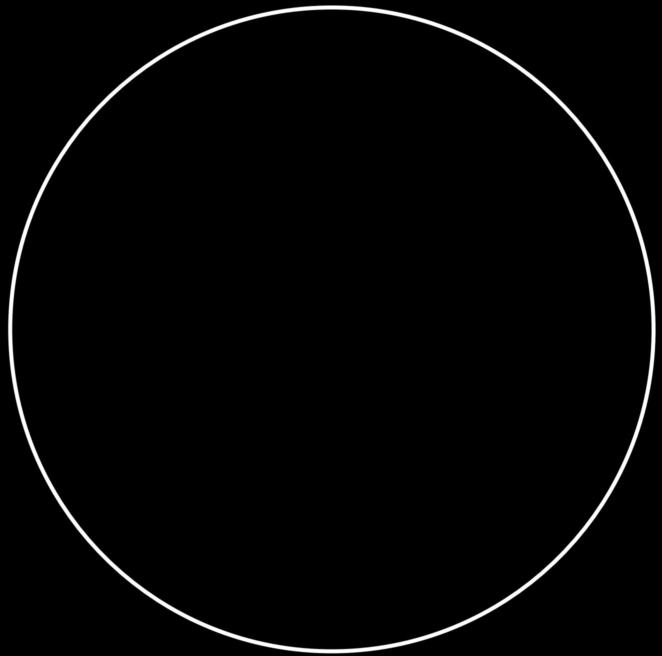
Start with non-Archimedean ordered field $\mathbb{R}(\varepsilon)$, where $\varepsilon > 0$ such that $\forall x \in \mathbb{R}_{>0} \colon \varepsilon < x$

tree

As a set

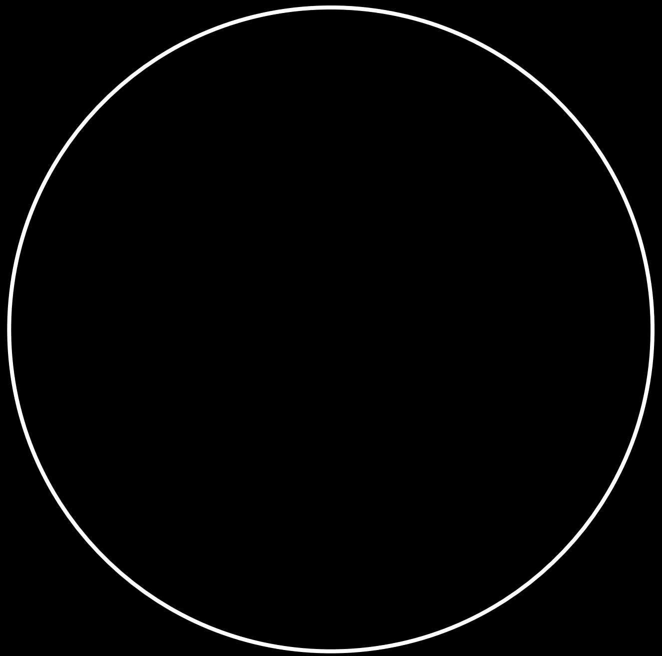
$\mathbb{R}$

$$\mathbb{H}^2 = \left\{ (x, y) \in \mathbb{R}^2 \colon \sqrt{x^2 + y^2} < 1 \right\}$$



$\mathbb{R}(\varepsilon) \rightsquigarrow$ Euclidean closure $\overline{\mathbb{R}(\varepsilon)}$

$$\mathbb{H}^2_\varepsilon = \left\{ (x, y) \in \overline{\mathbb{R}(\varepsilon)^2} \colon \sqrt{x^2 + y^2} < 1 \right\}$$



As a metric space

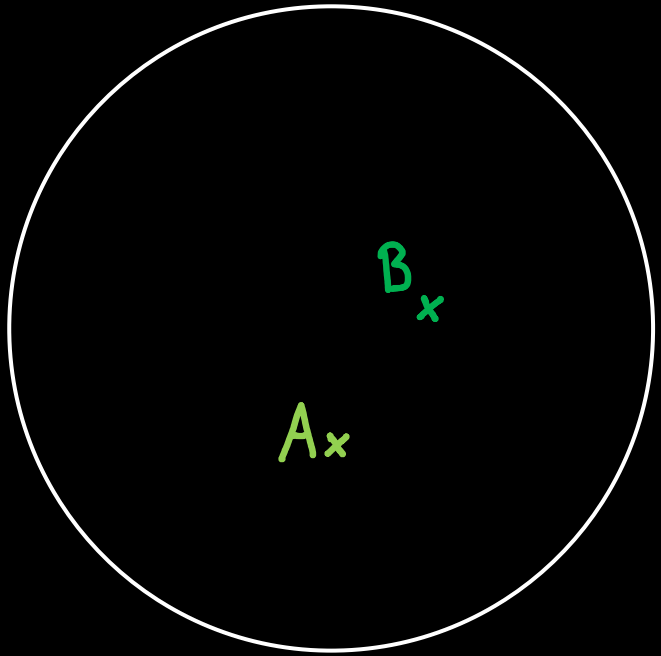
$\mathbb{R}$

$d(A, B) := \log \mathrm{cr}(A, B)$

<u>Prop:</u> $(\mathbb{H}^2, d)$
is a metric space.

$\mathbb{H}^2$ is the hyperbolic plane.

Crossratio: $cr(A, B)$



$\overline{\mathbb{R}(\varepsilon)}$

Valuation $v\colon \overline{\mathbb{R}(\varepsilon)} \to \mathbb{R}$

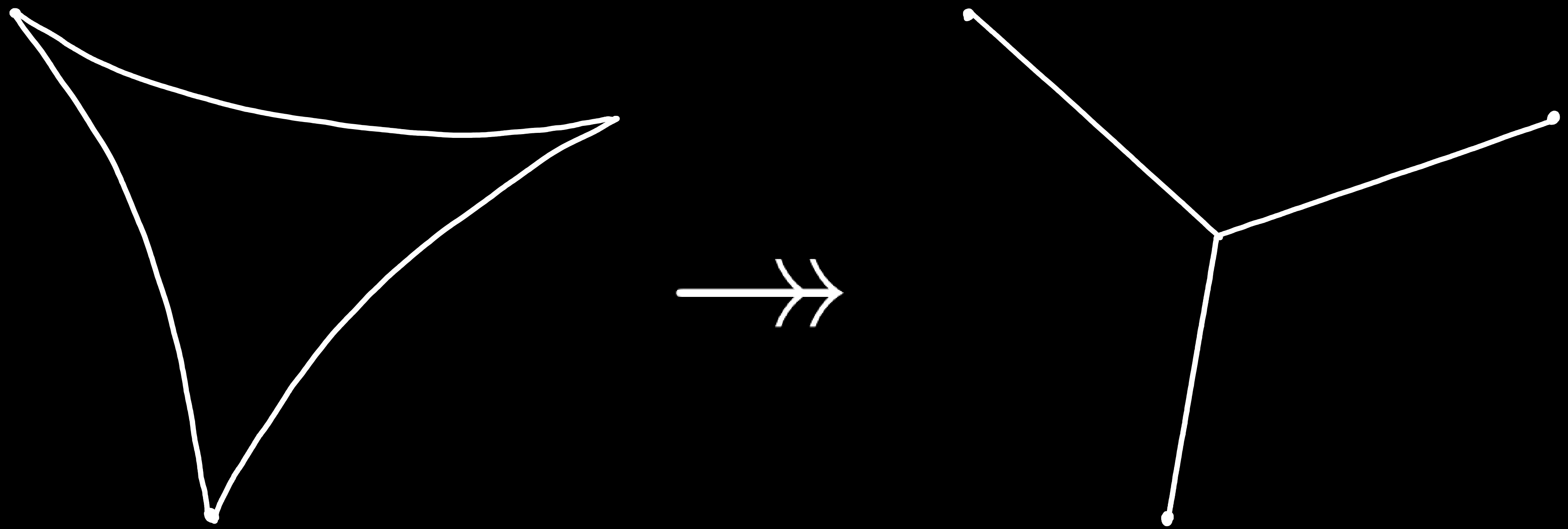
$d_\varepsilon(A, B) := v\, \mathrm{cr}(A, B)$

<u>Prop:</u> $(\mathbb{H}^2_\varepsilon, d_\varepsilon)$
is a semi-metric space.

<u>Cor:</u> $(\mathbb{H}^2_\varepsilon/\sim, d_\varepsilon)$
is a metric space.

How triangles look like

in $\mathbb{H}^2$ and $\mathbb{H}^2_\varepsilon$ in $\mathbb{H}^2_\varepsilon/\sim$



Prop: $\mathbb{H}^2_\varepsilon/\sim$ is a $\mathbb{R}$-tree.

Thanks for
your attention!

Source: G. W. Brumfiel, *The tree of a Non-Archimedean hyperbolic plane*, Contemporary Mathematics, Volume **74**, 1988.